

Why Does a Soccer Ball Have 12 Pentagons?

Topology and the Hidden Rules of Shape

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Why This Book?

To many people, mathematics is associated first and foremost with calculation: carrying out computations, manipulating formulas, and finding the correct answer. Yet much of mathematics looks quite different. Often, the hardest part is not the calculation itself but finding the right way to look at the problem.

I tend to think about mathematics geometrically. This is true even for problems that at first seem purely analytic and are usually expressed in formulas. A picture, a deformation of a surface, the image of a landscape, or the idea of moving through a space can suggest what ought to be true long before a formal proof appears.

Such intuition does not replace rigorous reasoning. In mathematics, it must eventually be checked, sharpened, and turned into proof. But it is often the first step toward understanding a problem. In this book, I therefore care less about collecting final answers than about understanding why things are the way they are—where a definition comes from, why a formula has the form it does, and what is really hiding behind a theorem.

Why does a soccer ball have twelve pentagons? Could inhabitants of a surface discover its curvature without ever seeing it from the outside? Why can't every landscape exist? Can a sphere be combed? Can every point be moved so that none remains where it started?

Sometimes the answer will require a formula, sometimes a picture, and sometimes only a simple thought experiment. The most important goal, however, is not to memorize the names of theorems or mathematical symbols.

If a mathematical term appears once and is not needed later, you

can safely forget it. If a more formal calculation is marked as something for interested readers, you can skip it and return to the main story. The illustrations are not here merely to decorate the text; quite often, they are meant to replace a definition or part of a proof.

What I would most like you to carry away from each chapter is an image and a question. Mathematics often begins with the right question.

In several places you will also find boxes titled *What is still unknown?*. They are there to remind us that we are not walking through a museum of a science that was completed long ago.

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